PHYSICAL REVIEW E VOLUME 60, NUMBER 6 DECEMBER 1999

Kinetic and transport equations for localized excitations in the sine-Gordon model

I. V. Baryakhtar
B. I. Verkin Institute for Low Temperature Physics and Engineering, Lenin Avenue 47, Kharkov 310164, Ukraine

V. G. Baryakhtar
Institute of Magnetism, 36b Vernadsky Street, Kiev 252142, Ukraine

E. N. Economou
Research Center of Crete, FORTH, P.O. Box 1527, 71110 Heraklion, Crete, Greece

(Received 6 April 1999)

We analyze the kinetic behavior of localized excitations—solitons, breathers, and phonons—in the Sine-Gordon model. Collision integrals for all types of localized excitation collision processes are constructed, and the kinetic equations are derived. We prove that the entropy production in the system of localized excitations takes place only in the case of inhomogeneous distribution of these excitations in real and phase space. We derive transport equations for soliton and breather densities, temperatures, and mean velocities, i.e., show that collisions of localized excitations lead to the creation of diffusion, thermoconductivity, and intrinsic friction processes. The diffusion coefficients for solitons and breathers, describing the diffusion processes in real and phase space, are calculated. It is shown that diffusion processes in real space are much faster than the diffusion processes in phase space. [S1063-651X(99)03811-8]

PACS number(s): 05.20.Dd

I. INTRODUCTION

The problem of kinetic properties of excitations in integrable models belongs to a class of the most nontrivial problems of physical kinetics. First, enormously long relaxation of nonlinear excitation has been found in numerical experiments by Fermi, Pasta, and Ulam [1]. In Zabusky and Kruskal's numerical experiment [2] unexpected behavior of localized nonlinear excitations was discovered, namely their interaction without changing their forms and velocities. Zabusky and Kruskal named them solitons. Shortly after, the analytical method of solving nonlinear differential equation with partial derivatives—the inverse scattering method—was found in the framework of the Korteweg–de Vries (KdV) equation for $1+1$ dimensions [3]. Other physically reasonable continuous models, treatable by the inverse scattering method are the nonlinear Schrödinger (NS) equation, the Sine-Gordon (SG) equation, and the Landau-Lifshitz (LL) equation (see [4,5]). Common excitations in integrable models are: the one-parameter localized wave, i.e., the soliton and the nonlinear periodic wave for the KdV model; the two-parameter localized wave, i.e., the breather for the SG and LL models; and the envelope soliton for the NS equation. All these excitations interact without changing their forms, their velocities, and therefore their energies; the only result of their interaction is the shifts of their coordinates, and for breathers and NS solitons the change of their phases as well. It is worthwhile emphasizing that many-particle effects are absent in the following sense: the total shift in collisions involving several excitations is equal to the sum of the shifts in each collision (see [4]).

Soon after analytically solving the KdV equation [3], Zakharov considered the possibility of the kinetic decryption of solitons [6]. In [6], the kinetic equation for solitons was written, but only the effect of the renormalization of the soliton velocity due to soliton-soliton collisions has been considered. Another approach to the investigation of the kinetic properties of kink-type solitons was proposed in [7–9]. In [7] (numerically) and in [8,9] (analytically) the diffusion of kinks interacting with phonons in the ϕ^4 model was considered. This model is not exactly integrable, but the potential for a kink-phonon interaction has a reflectionless form and the result of the interaction is the same as for soliton-soliton collision in integrable models. Taking into account such kinds of interactions in [8,9], the kink diffusion coefficient was calculated. A bit later the same calculations were done for the SG model [10]. The renormalization of soliton velocity and the diffusion coefficient of the solitons due to soliton-soliton and soliton-magnon collisions have been considered for the SG model in [11].

To explain the effect of velocity renormalization it is useful to note that the dependence of the soliton coordinate x on time can be written in the following form:

$$x(t)=x_0+v_1t+\int_{-\infty}^{t}dt'\int dx_2dv_2 \times|v_1-v_2|\Delta x_s(v_1,v_2)\,\theta(t-t')f(x_2,v_2,t'), \tag{1.1}$$

when shifts of soliton coordinates $\Delta x_s(v_1,v_2)$ are taken into consideration. In formula (1.1) $f(x,v,t)$ is the distribution function of solitons or phonon (magnons), and the sample soliton collides with the soliton or phonon (magnon) wave packet (x_2,v_2) at the moment t',

$$\theta(t)=1,\ \ t>0,\qquad \theta(t)=0,\ \ t<0. \tag{1.2}$$

From formula (1.1) one can obtain the following expression for the average velocity of the sample soliton:

$$\langle v_1\rangle = v_1 + \int dx_2 dv_2 |v_1 - v_2| \Delta x_s(v_1, v_2) f(x_2, v_2, t). \tag{1.3}$$

The diffusion coefficient of soliton is given by

$$D = \frac{\langle [x(t) - \langle v\rangle t - x(0)]^2\rangle}{2t} \tag{1.4}$$

(see [8–10]).

So the shift of the soliton coordinate Δx leads to its diffusion. Do other kinetic coefficients for solitons exist in the integrable models? This question is closely related to the problem of entropy production. From the fact of conserving the soliton velocities follows the absence of entropy production in momentum space only. But in the coordinate space the interaction of the solitons is very strong. Therefore the question, does entropy increase in integrable systems, is not nonsensical. The answer to these questions has been given in [12] for the example of kink-type solitons in the SG model. In [12] the kinetic equation for the gas of kinks has been formulated, the entropy production has been proved, and the existence of the coefficients of intrinsic friction and thermal conductivity has been shown. Obviously such kinds of kinetic coefficients as the mobility coefficient appear in the presence of perturbations, which destroy the integrability of the model (see [13] and a review [14]).

In this paper the kinetic behavior of solitons, breathers, and phonons in the framework of the SG model is considered. We construct collision integrals for all possible collisions and formulate the system of Boltzmann-type kinetic equations for solitons, breathers, and phonons. Based on the system of kinetic equations thus obtained, we prove that the entropy production takes place as a result of randomization of the distribution of excitations in coordinate and phase space. Thus we are able to derive transport equations for solitons and breathers and to calculate self-diffusion coefficients for soliton-soliton, soliton-breather, and breather-breather collisions as an example.

II. ELEMENTARY EXCITATIONS IN THE SINE-GORDON EQUATION

The Sine-Gordon equation in dimensionless variables can be written as

$$\varphi_{tt} - \varphi_{xx} + \frac{m^2}{\beta}\sin\beta\varphi = 0. \tag{2.1}$$

Here φ is the order parameter, t is the dimensionless time, x is the dimensionless space coordinate, $c=1$ is the characteristic velocity, m is the dimensionless mass, and β is the parameter of nonlinearity.

Equation (2.1) follows from Hamilton's equations (see, for example, [5])

$$\varphi_t = \{H, \varphi\}, \quad \pi_t = \{H, \pi\}, \tag{2.2}$$

with the Hamiltonian

$$H = \frac{1}{2}\int \left[\pi^2 + \varphi_x^2 + 2\left(\frac{m}{\beta}\right)^2 (1 - \cos\beta\varphi)\right] dx \equiv \int h(\varphi, \pi). \tag{2.3}$$

In Eqs. (2.2) and (2.3) φ and π are the canonically conjugated coordinate and momentum:

$$\{\pi(x), \varphi(y)\} = \delta(x-y), \tag{2.4}$$

where the Poisson brackets are defined in the usual way:

$$\{A, B\} = \int \left(\frac{\delta A}{\delta\pi(x)}\frac{\delta B}{\delta\varphi(x)} - \frac{\delta A}{\delta\varphi(x)}\frac{\delta B}{\delta\pi(x)}\right) dx. \tag{2.5}$$

It is easy to see that the functionals of total momentum P of the system

$$P = -\int \pi(x,t)\frac{\partial\varphi(x,t)}{\partial x} dx \tag{2.6}$$

and total ''angular'' momentum K

$$K = \int x h[\pi, \varphi] dx, \tag{2.7}$$

commute with the Hamiltonian (2.3).

It is well known that there are two types of localized excitations (LEs) in the SG model: solitons and breathers. The soliton is a one-parameter solution, and the breather is a two-parameter solution; the breather can be interpreted as the bound states of two solitons (see [4,5]). The one-parameter solution of the SG model, often named the kink, can be written as

$$\varphi_s = -4\frac{\epsilon}{\beta}\arctan\{\exp[\delta_s^{-1}(x - v_s t - x_{0s})]\}. \tag{2.8}$$

The breather solution has the following form:

$$\varphi_b = \frac{4}{\beta}\arctan\left(\frac{\omega_2}{\omega_1}\right)\frac{\sin[\omega(v_b)t - k(v_b)x - \varphi_{0b}]}{\cosh[(\delta_b^{-1}(x - v_b t - x_{0b})]}, \tag{2.9}$$

where

$$m(v) = m/\sqrt{1-v^2}, \quad \omega(v) = m(v)\omega_1, \quad k(v) = v m(v)\omega_1,$$
$$\delta_s^{-1} = m(v_s), \quad \delta_b^{-1} = \omega_2 m(v_b). \tag{2.10}$$

Here $\epsilon = \{+1, -1, 0\}$ is the soliton topological charge, $+1$ corresponds to the soliton, -1 to the antisoliton, 0 to the breather, v_s and v_b are the soliton and breather velocities, δ_s and δ_b are the soliton and breather sizes, x_{0s} and x_{0b} are the initial coordinates of the soliton and breather, φ_{0b} is the initial breather phase, and ω_1 and ω_2 are the breather parameters satisfying the following condition (see [5]):

$$\omega_1^2 + \omega_2^2 = 1. \tag{2.11}$$

When $\omega_2 \to 0$, the breather solution reduces to the following:

$$\lim_{\omega_2\to 0} \varphi_b(x,t)=\varphi_p(x,t)$$

$$=4\frac{\omega_2}{\beta}\frac{\sin[\omega(v)t-k(v)x-\varphi_{0b}]}{\cosh[m(v)\omega_2(x-kt/\omega-x_{0b})]}, \tag{2.12}$$

and gives a plane-wave solution, i.e., a phonon.

The phonon frequency is related to the wave vector **k** by the formula

$$\omega_{ph}^2(k)=m^2+k^2. \tag{2.13}$$

The breather velocity, when $\omega_2\to 0$, reduces to the phonon group velocity:

$$v_{ph}=(k/\omega_{ph})=(\partial\omega_{ph}/\partial k), \tag{2.14}$$

and $4\omega_2$ becomes the amplitude of phonon oscillations.

From formulas (2.8), (2.9), and (2.3) it is easy to find the soliton and the breather energy:

$$E_s=M_s/\sqrt{1-v_s^2},\quad E_b=M_b/\sqrt{1-v_b^2}. \tag{2.15}$$

The soliton and the breather rest masses are given by

$$M_s=8m,\quad M_b=16m\omega_2. \tag{2.16}$$

When $\omega_2\to 1$, the breather reduces to the soliton-antisoliton bound state with a mass $16m$. The difference between the breather energy and the sum of energies of the isolated soliton and the antisoliton gives the binding energy of the breather:

$$\Delta E=-\frac{16m}{\sqrt{1-v^2}}(1-\omega_2). \tag{2.17}$$

If the system consists of n_1 solitons and an n_2 breather, we have

$$E=\sum_{s=1}^{n_1}E_s+\sum_{b=1}^{n_2}E_b,\quad P=\sum_{s=1}^{n_1}P_s+\sum_{b=1}^{n_2}P_b,$$

$$K=\sum_{s=1}^{n_1}K_s+\sum_{b=1}^{n_2}K_b. \tag{2.18}$$

Energy and momentum are related by following formula:

$$E_i^2=M_i^2+P_i^2. \tag{2.19}$$

Formulas (2.18) allow us to consider solitons and breathers as elementary excitations of field φ.

III. COLLISIONS OF LOCALIZED EXCITATIONS

Due to the one-dimensionality of the problem the collisions of solitons and breather take place without dependence on the values x_{0s} and x_{0b}, $s=1,...,n_1$, $b=1,...,n_2$, because their velocities do not change. This peculiar property follows from energy- and momentum-conservation laws in the Sine-Gordon equation as well as other exactly integrable models (see [4,5]). Those conservation laws have the form

$$v_1=v_1',\quad v_2=v_2'. \tag{3.1}$$

Here and later the values after collision will be denoted with a prime.

To analyze collision processes it is necessary to take into account the the ‘‘angular’’-momentum conservation law, which can be written as

$$x_1E_1+x_2E_2=x_1'E_1'+x_2'E_2'. \tag{3.2}$$

The result of collision is coordinate shifts (changes), and for breathers phase shifts as well. Furthermore, only pair collisions exist and there are not many particles effects (see [4]). Therefore formulas (3.1) and (3.2) provide the general framework for studying the effects of collisions. In this section conservation laws and the coordinate x_{01} and phase changes φ_{01} for all types of collisions will be written explicitly from general formulas (see [4,5]). For simplicity the index 0 will be omitted.

A. Soliton-soliton collisions

Conservation laws for the two-soliton collision have the following form:

$$v_{1s}'=v_{1s},\quad v_{2s}'=v_{2s},\quad (x_{1s}'-x_{1s})E_{1s}+(x_{2s}'-x_{2s})E_{2s}=0. \tag{3.3}$$

Coordinates shifts are given by

$$\Delta x_{1s}=x_{1s}'-x_{1s}=\frac{4}{E_{1s}}\operatorname{sgn}(v_0)\ln|Z_{ss}|=\frac{\delta_{1s}}{2}\operatorname{sgn}(v_0)\ln|Z_{ss}|,$$

$$\Delta x_{2s}=x_{2s}'-x_{2s}=-\frac{4}{E_{2s}}\operatorname{sgn}(v_0)\ln|Z_{ss}|$$

$$=-\frac{\delta_{2s}}{2}\operatorname{sgn}(v_0)\ln|Z_{ss}|, \tag{3.4}$$

where

$$Z_{ss}=\frac{1+\sqrt{1-v_0^2}}{1-\sqrt{1-v_0^2}},\quad v_0=\frac{(v_{1s}-v_{2s})}{(1-v_{1s}v_{2s})}. \tag{3.5}$$

Here v_0 is the velocity of the relative motion of solitons, which in the general case can be written as

$$v_0=\frac{(v_{1i}-v_{2k})}{(1-v_{1i}v_{2k})},\quad i,k=s,b,ph. \tag{3.6}$$

B. Soliton-breather collision

In this case conservation laws can be written as

$$v_{1s}=v_{1s}',\quad v_{2b}=v_{2b}',\quad \omega_{2,b2}=\omega_{2,b2}',$$
$$(x_{1s}'-x_{1s})E_{1s}+(x_{2b}'-x_{2b})E_{2b}=0. \tag{3.7}$$

Coordinate shifts Δx are described by the following formulas:

$$\Delta x_{1s}=x_{1s}'-x_{1s}=\frac{8}{E_{1s}}\operatorname{sgn}(v_0)\ln|Z_{sb}|=\delta_{1s}\operatorname{sgn}(v_0)\ln|Z_{sb}|,$$

$$\Delta x_{2b}=x'_{2b}-x_{2b}=-\frac{8}{E_{2b}}\,\mathrm{sgn}(v_0)\ln|Z_{sb}|$$
$$=-\frac{\delta_{2s}}{2}\,\mathrm{sgn}(v_0)\ln|Z_{sb}|, \qquad (3.8)$$

where

$$Z_{sb}=\frac{1+\omega_{2,b2}\sqrt{1-v_0^2}}{1-\omega_{2,b2}\sqrt{1-v_0^2}}, \quad v_0=\frac{(v_{1s}-v_{2b})}{(1-v_{1s}v_{2b})}. \qquad (3.9)$$

The breather phase shift $(\Delta\varphi)_b$ is determined by the formula

$$\tan(\Delta\varphi)_{b2}=-\mathrm{sgn}(v_0)\,\frac{\sqrt{1-v_0^2}\,\omega_{1,b2}}{v_0}. \qquad (3.10)$$

Some comments regarding formulas (3.8) and (3.10) are in order. For the soliton coordinate shift [formula (3.8)] due to the soliton, the soliton-breather collision one has a factor of 8 instead of 4 as in formula (3.4). This difference can be explained as follows. If $\omega_{2,b2}\rightarrow 1$, one can consider a breather as two solitons with opposite topological charge. Since the coordinate shift does not depend on topological charge, the soliton-breather collision in this case can be considered as a soliton–two-soliton collision, and thus the shift is doubled. Let us mention that in the case $\omega_{2,b2}\rightarrow 1$, the quantities Z_{ss} and Z_{sb} are the same.

Formula (3.10) implies that the phase shift in general is large: $(\Delta\varphi)_{b2}\sim\pi$. It is small only in the ultrarelativistic case $|v_0-1|\ll 1$. In this case $(\Delta\varphi)_{b2}\approx-\mathrm{sgn}(v_0)\sqrt{1-v_0^2}\,\omega_{1,b2}$. If $v_0\ll 1$, then $(\Delta\varphi)_{b2}\approx\pi/2$.

C. Soliton-phonon collision

This process is characterized by the following conservation laws:

$$v'_{1s}=v_{1s}, \quad v'_{2ph}=v_{2ph}, \quad \omega'_{ph}=\omega_{ph},$$
$$(x'_{1s}-x_{1s})E_{1s}+(x'_{2ph}-x_{2ph})E_{2ph}=0. \qquad (3.11)$$

The expressions for x and φ shifts are

$$\Delta x_{2ph}=x'_{2ph}-x_{2ph}=-\mathrm{sgn}(v_0)\,\frac{m\sqrt{1-v_s^2}}{\omega_{ph}(\omega_{ph}-kv_s)}, \qquad (3.12)$$

$$\Delta x_{1s}=x'_{1s}-x_{1s}=-\frac{E_{2ph}}{E_{1s}}\Delta x_{2ph}, \qquad (3.13)$$

$$\tan(\Delta\varphi_{2ph})=-\mathrm{sgn}(v_0)\,\frac{\sqrt{1-v_0^2}}{v_0}. \qquad (3.14)$$

Phonon energy E_{2ph} and frequency $\omega_{ph}(k)$ are defined by the formulas

$$E_{2ph}=16\omega_2\omega_{ph}(k), \quad \omega_{ph}^2(k)=m^2+k^2. \qquad (3.15)$$

D. Breather-breather collisions

The corresponding conservation laws have the form

$$v'_{1b}=v_{1b}, \quad v'_{2b}=v_{2b}, \quad \omega'_{2,b1}=\omega_{2,b1}, \quad \omega'_{2,b2}=\omega_{2,b2}, \qquad (3.16)$$
$$(x'_{1b}-x_{1b})E_{1b}+(x'_{2b}-x_{2b})E_{2b}=0.$$

The expressions for the x_b and φ shifts are

$$\Delta x_{1b}=\frac{8}{E_{1b}}\,\mathrm{sgn}(v_0)\ln|Z_{bb}X'_{bb}|=\frac{\delta_{1b}}{2}\,\mathrm{sgn}(v_0)\ln|Z_{bb}Z'_{bb}|,$$

$$\Delta x_{2b}=-\frac{8}{E_{2b}}\,\mathrm{sgn}(v_0)\ln|Z_{bb}Z'_{bb}|=-\frac{\delta_{2b}}{2}\,\mathrm{sgn}(v_0)\ln|Z_{bb}Z'_{bb}|, \qquad (3.17)$$

$$\tan(\Delta\varphi_{1b})=-\mathrm{sgn}(v_0)\,\frac{2v_0\sqrt{1-v_0^2}\sin\psi_2\cos\psi_1}{v_0^2+(1-v_0^2)(\cos^2\psi_2-\cos^2\psi_1)}, \qquad (3.18)$$

$$\tan(\Delta\varphi_{2b})=-\mathrm{sgn}(v_0)\,\frac{2v_0\sqrt{1-v_0^2}\sin\psi_1\cos\psi_2}{v_0^2+(1-v_0^2)(\cos^2\psi_1-\cos^2\psi_2)}. \qquad (3.19)$$

Here

$$Z_{bb}=\frac{1-\sqrt{1-v_0^2}\cos(\psi_1+\psi_2)}{1-\sqrt{1-v_0^2}\cos(\psi_1-\psi_2)},$$
$$Z'_{bb}=\frac{1+\sqrt{1-v_0^2}\cos(\psi_1-\psi_2)}{1+\sqrt{1-v_0^2}\cos(\psi_1+\psi_2)}. \qquad (3.20)$$

The angles ψ_1 and ψ_2 are related to the parameters $\omega_{2,b1}$ and $\omega_{2,b2}$ by the formulas

$$\sin\psi_1=\omega_{2,b1}, \quad \sin\psi_2=\omega_{2,b2}. \qquad (3.21)$$

E. Breather-phonon collision

Since phonons are the limiting case of the breather when $\omega_2\rightarrow 0$, formulas for the breather-phonon collision can be obtained from ones in the preceding subsection by passing to the limit $\omega_{2,b2}\rightarrow 0$. This means that the second breather reduces to a phonon wave packet and the notion of group velocity, and the coordinate of the wave packet automatically appears. The breather velocity v_b reduces to the group velocity of the wave packet and the coordinate x_b reduces to the coordinate of the center of the wave packet.

The corresponding conservation law has the form

$$v'_{1b}=v_{1b}, \quad v'_{2ph}=v_{2ph}, \quad \omega'_{2,b1}=\omega_{2,b1}, \quad \omega'_{2,ph2}=\omega_{2,ph2}, \qquad (3.22)$$
$$(x'_{1b}-x_{1b})E_{1b}+(x'_{2ph}-x_{2ph})E_{2ph}=0,$$

where

$$\Delta x_{2ph}=-\mathrm{sgn}(v_0)\,\frac{2m\omega_{2,b1}\sqrt{1-v_0^2}}{\omega_{ph}[\omega_{ph}-kv_b][1-(1-v_0^2)\omega_{1,b1}^2]}, \qquad (3.23)$$

$$\Delta x_{1b}=-\frac{E_{2ph}}{E_{1b}}\Delta x_{2ph}$$

$$\Delta\varphi_{1b} = -\operatorname{sgn}(v_0)\,\frac{2v_0\sqrt{1-v_0^2}\,\omega_{1,b1}\omega_{2,ph2}}{v_0^2+(1-v_0^2)\,\omega_{2,b1}^2}, \qquad (3.24)$$

$$\tan(\Delta\varphi_{2ph}) = -\operatorname{sgn}(v_0)\,\frac{2v_0\sqrt{1-v_0^2}\,\omega_{2,b1}}{v_0^2-(1-v_0^2)\,\omega_{2,b1}^2}. \qquad (3.25)$$

F. Phonon-phonon collisions

As is well known, in linear theory phonon-phonon collisions are absent. Here the phonon-phonon interaction is the result of the nonlinearity of the Sine-Gordon equation. Corresponding relations can be obtained from ones in the preceding subsection by passing the limit $\omega_{2,b1}\to 0$. By considering $\omega_{2,b1}$ as a small quantity, $\omega_{2,b1}\ll 1$, one has

$$v'_{1ph}=v_{1ph}, \quad v'_{2ph}=v_{2ph}, \quad \Delta x_{1ph}E_{1ph}+\Delta x_{2ph}E_{2ph}=0. \qquad (3.26)$$

The shift of the phonon's coordinate is

$$\Delta x_{1ph} = \operatorname{sgn}(v_0)\,\frac{32}{E_{1ph}}\,\frac{\sqrt{1-v_0^2}\,\omega_{2,ph2}\omega_{2,ph1}}{v_0^2},$$

$$\Delta x_{2ph} = -\frac{E_{1ph}}{E_{2ph}}\,\Delta x_{1ph}, \qquad (3.27)$$

where

$$E_{1ph}=16\omega_{2,ph1}\omega_{ph1}, \quad E_{2ph}=16\omega_{2,ph2}\omega_{ph2}. \qquad (3.28)$$

The change of phonon phase is

$$\Delta\varphi_{1ph} = -2\operatorname{sgn}(v_0)\,\omega_{2,ph2}\frac{\sqrt{1-v_0^2}}{v_0},$$

$$\Delta\varphi_{2ph} = -2\operatorname{sgn}(v_0)\,\omega_{2,ph1}\frac{\sqrt{1-v_0^2}}{v_0}. \qquad (3.29)$$

The examination of phonons as a limiting case of breathers offers the possibility of considering changes in phonon phase and coordinate simultaneously.

Let us analyze the phase shifts of breathers and phonons. It is easy to see from formulas (3.10) and (3.14) that breather and phonon shifts are not small for soliton-breather and soliton-phonon collisions in the general case. For breather-phonon collisions, breather and phonon shifts are essentially different [see Eqs. (3.24) and (3.25)]. Breather phase shifts are proportional to ω_{2ph} and therefore small. The phonon phase shift $(\Delta\varphi)_{ph}\approx 1$. For phonon-phonon collisions, phase shifts are small [see Eq. (3.29)]. This agrees with the standard linear theory of phonons. In accordance with the usual theory of phonons, shifts of phonon wave packets do not take place.

In conclusion, let us emphasize that the soliton topological charge is conserved for each type of collision. The processes leading to soliton-antisoliton bound-state formation and the reverse processes do not take place because of energy conservation. Thus in the framework of the exactly integrable Sine-Gordon model with Hamiltonian (2.3) the numbers of solitons, antisolitons, breathers, and phonons are defined by initial conditions at each point of (x,t). Interactions destroying the integrability of the system lead to the possibility of energy exchange between the LE and to the formation and decay of breathers. In the present paper only the processes with conservation of energy and topological charge have been considered. The processes with energy exchange will be analyzed in a future publication.

IV. PROBABILITY OF SCATTERING PROCESSES

To construct collision integrals for processes considered in this section let us analyze the probability of the corresponding scattering process. The simplest process is soliton-soliton scattering. By definition the initial state i of a soliton pair is described by its velocities and center-of-gravity coordinates:

$$i\equiv(x_1,v_1,x_2,v_2)\equiv(1,2). \qquad (4.1)$$

The final state f is defined by

$$f\equiv(x'_1,v'_1,x'_2,v'_2)\equiv(1',2'). \qquad (4.2)$$

The current density of solitons per unit density is given by

$$j_1=|v_0|. \qquad (4.3)$$

Taking into account formula (4.3) and the fact that two solitons collide in any case, the transition probability per unit time from state i to state f can be presented as

$$\begin{aligned} W_{i\to f}&\equiv W(1',2'|1,2)\\ &=|v_0|\,\delta(v'_1-v_1)\,\delta(v'_2-v_2)\\ &\quad\times\delta\!\left(\frac{E_1}{E_2}(x'_1-x_1)+(x'_2-x_2)\right)\\ &\quad\times\delta(x'_1-x_1-\Delta x(v_1,v_2)). \end{aligned} \qquad (4.4)$$

In this formula the first two δ functions describe energy conservation laws, the third one the ‘‘angular’’-momentum conservation law, and the last describes the coordinate shift. The coordinate shift of the second soliton follows from the angular-momentum conservation law. The coordinate shift Δx_1 is defined by expression (3.4), (3.5).

It is convenient to rewrite formula (4.4) in the form

$$\begin{aligned} W(1',2'|1,2)&=R_{ss}(v'_1,v'_2|v_1,v_2)\,\delta(x'_1-x_1\\ &\quad-\Delta x_1(v_1,v_2))\,\delta(x'_2-x_2-\Delta x_2(v_1,v_2)). \end{aligned} \qquad (4.5)$$

where

$$R_{ss}(v'_1,v'_2|v_1,v_2)=|v_0|\,\delta(v'_1-v_1)\,\delta(v'_2-v_2). \qquad (4.6)$$

Let us note that

$$R_{ss}(v'_1,v'_2|v_1,v_2)=R_{ss}(v_1,v_2|v'_1,v'_2)=R_{ss}(v'_2,v'_1|v_2,v_1). \qquad (4.7)$$

For the probability of the inverse process the following expression can be written:

$$W(1,2|1',2')=R_{ss}(v_1',v_2'|v_1,v_2)\,\delta(x_1-x_1'+\Delta x_1(v_1,v_2))\,\delta(x_2-x_2'+\Delta x_2(v_1,v_2)). \tag{4.8}$$

Let us emphasize that the probability defined by formula (4.4) does not satisfy the detailed balance principle in the standard form

$$\tilde{W}(1',2'|1,2)=\tilde{W}(1^*,2^*|1'^*,2'^*). \tag{4.9}$$

In Eq. (4.9) the following notation has been used: $(1^*)\equiv(x_1,-v_1)$.

For further analysis it is convenient to present the expression (4.5) for the probability W as a sum of two parts W^r and W^m:

$$W_{ss}(1,2|1',2')=W^r_{ss}(1,2|1',2')+W^m_{ss}(1,2|1',2'), \tag{4.10}$$

where

$$W^r_{ss}=\frac{1}{2}R_{ss}[\delta(x_{1s}-x_{1s}'-\Delta x_{1s})\,\delta(x_{2s}-x_{2s}'-\Delta x_{2s})-\delta(x_{1s}-x_{1s}'+\Delta x_{1s})\,\delta(x_{2s}-x_{2s}'+\Delta x_{2s})] \tag{4.11}$$

$$W^m_{ss}=\frac{1}{2}R_{ss}[\delta(x_{1s}-x_{1s}'-\Delta x_{1s})\,\delta(x_{2s}-x_{2s}'-\Delta x_{2s})+\delta(x_{1s}-x_{1s}'+\Delta x_{1s})\,\delta(x_{2s}-x_{2s}'+\Delta x_{2s})]. \tag{4.12}$$

The probability W^r_{ss}, as will be shown later, describes the effect of the *renormalization* of the soliton velocity, and the probability W^m_{ss} describes the homogenization of the soliton phenomena, i.e., the effect of mixing states. The quantity W^m_{ss} is related to entropy production in the soliton gas and the kinetic coefficients (see Secs. V and VI).

It is easy to convince oneself that

$$W^m_{ss}(1',2'|1,2)=W^m_{ss}(1,2|1',2'). \tag{4.13}$$

This equality means that for the dissipative part of the transition probability the usual detailed balance principle is valid.

Let us consider now the general case of the LE i and k $(i,k=s,b,ph)$ for soliton, breather, and phonon correspondingly. The probability of such types of collisions per unit of time W_{ik} can be defined by following formula:

$$W_{ik}\equiv W_{ik}(1',2'|1,2)=R_{ik}\delta(X_i'-X_i-d_i(1,2))\,\delta(X_k'-X_k-d_k(1,2)), \tag{4.14}$$

where

$$R_{ik}=R_{ik}(V_i',V_k'|V_i,V_k)=|v_0|\,\delta(V_i-V_i')\,\delta(V_k-V_k'). \tag{4.15}$$

In formulas (4.14) and (4.15) the following notations have been used. Numbers 1 and 2 mean the set of variables defining the colliding LE states; d_i is the shift of the LE coordinate or phase. In the case of solitons, $1\equiv(x_s,v_s)$; for the breather or phonon, $1\equiv(x_i,\varphi_i,v_i,\omega_{2i})$, $i=b,ph$. For simplicity the two-component coordinate X_i and the two-component velocity V_i have been introduced in the following way:

$$X_s=x_s,V_s=v_s,\quad X_i=(x_i,\varphi_i),\quad V_i=(v_i,\omega_{2i}),\quad i=b,ph. \tag{4.16}$$

In Eq. (4.15) the delta function $\delta(V_i-V_i')$ describes the conservation law of energy and the momentum of each LE; in the expression (4.14), the delta function $\delta(X_i'-X_i-d_i(1,2))$ defines the coordinate and phase for breather and phonon shifts after collision.

From the definition of R_{ik} it follows that

$$R_{ik}=R(V_i',V_k'|V_i,V_k)=R(V_i,V_k|V_i',V_k'). \tag{4.17}$$

Later only the case of a weak inhomogeneous gas of LE will be considered; i.e., when the characteristic length l_i of the distribution function is much larger than the shift ΔX_i,

$$|l_i|\gg|\Delta X_i|. \tag{4.18}$$

The mean-free path of the LE is

$$l_i\sim\frac{1}{n_i}, \tag{4.19}$$

because the cross section of the scattering is equal to 1.

Using the expression for the breather coordinate shift (see Sec. III), the strong inequality (4.18) can be rewritten as

$$\frac{8}{E_b}\ln|Z_{bb}Z_{bb}'|\ll l_i \tag{4.20}$$

(kink and phonon cases can be obtained putting $\omega_2\to1$ and $\omega_2\to0$ correspondingly). If the relative velocity v_0 is not small, the condition (4.20) becomes

$$m(v)\omega_2\gg\frac{1}{l_i}\sim n_i,\quad i=s,b,ph. \tag{4.21}$$

This means that average distance between the LE is much larger than their sizes. In other words, the condition of small inhomogeneous Eq. (4.18) coincides with the condition of small density of the LE gas.

When $v_1\approx v_2$, and if for simplicity the case $\omega_{21}=\omega_{22}=\omega_0$ is considered, the strong inequality (4.20) leads to the following one:

$$1\gg v_0^2\gg\exp^{-l_i m(0)\omega_0/8}. \tag{4.22}$$

The condition (4.22) means that the region of velocities, where the gas approximation cannot be satisfied, is exponentially small. Moreover, if LEs are distributed with some function in coordinate space (see the following sections), the contribution of such types of excitations in LE kinetics is small.

Let us present W_{ik} as a sum of two parts W^r_{ik} and W^m_{ik} in the same way as Eqs. (4.10)–(4.12). Assuming that Eq. (4.18) is valid, we can expand the δ functions in the expression (4.14) for $W_{ik}(1,2|1,2)$ in power series ΔX_i. Then the expressions for W_{ik}^r and W_{ik}^m become

$$W_{ik}^r(1',2'|1,2)=R_{ik}(1'2',|1,2)\left(d_i\frac{\partial}{\partial X_i}+d_k\frac{\partial}{\partial X_k}\right)\times\delta(X_i-X_i')\,\delta(X_k-X_k'),\qquad(4.23)$$

$$W_{ik}^m(1',2'|1,2)=R_{ik}(1'2',|1,2)\left[2+d_i^2\frac{\partial^2}{\partial X_i^2}+d_k^2\frac{\partial^2}{\partial X_k^2}+2d_id_k\frac{\partial^2}{\partial X_i\partial X_k}\right]\delta(X_i-X_i')\,\delta(X_k-X_k').\qquad(4.24)$$

V. COLLISION INTEGRALS

In this section the expressions for the collision integrals for all types of LE in the SG equation will be derived. It is easier to do this starting from the formulas for the probability of collisions. Let us introduce following notations:

$$f=f(X,V,t),\quad B=B(X,V,t),\quad N=N(X,V,t)\qquad(5.1)$$

for the distribution functions of solitons, breathers, and phonons *f*, *B*, and *N*, respectively. The probability of finding the LE in the state $(1,1+d1)$ can be defined in the usual way:

$$dW_i=F_i(1)d1,\qquad(5.2)$$

where $i=s,b,ph$ and $F_s=f,F_b=B,F_{ph}=N$. The total result of the collisions of a sample LE with the other LE is the sum of each collision. This is due to the special type of interaction in exactly integrable models. Therefore the general collision integral can be presented as a sum of partial collision integrals. For example, in the case of solitons, the collision integral can be written as

$$\mathcal{L}=\mathcal{L}_{ss}\{f,f\}+\mathcal{L}_{sb}\{f,B\}+\mathcal{L}_{sph}\{f,N\},\qquad(5.3)$$

where $\mathcal{L}_{ss},\mathcal{L}_{sb},\mathcal{L}_{sph}$ are soliton-soliton, soliton-breather, and soliton-phonon collision integrals, respectively, which have the following form:

$$\mathcal{L}_{ss}\{f,f\}=\int d1'd2'd2\{W_{ss}(1,2|1',2')f(1')f(2')-W_{ss}(1',2'|1,2)f(1)f(2)\},\qquad(5.4)$$

$$\mathcal{L}_{sb}\{f,B\}=\int d1'd2'd2\{W_{sb}(1,2|1',2')f(1')B(2')-W_{sb}(1',2'|1,2)f(1)B(2)\},\qquad(5.5)$$

$$\mathcal{L}_{sph}\{f,N\}=\int d1'd2'd2\{W_{sph}(1,2|1',2')f(1')N(2')-W_{ss}(1',2'|1,2)f(1)N(2)\}.\qquad(5.6)$$

The first term in formula (5.4) and in formulas (5.5) and (5.6) describes solitons ‘‘arriving’’ at the state (1) as the result of collisions, and the second term describes solitons ‘‘leaving’’ this state.

The general expression for the collision integral has the form

$$\mathcal{L}_i=\sum_k\mathcal{L}_{ik}\{F_iF_k\}.\qquad(5.7)$$

Here

$$\mathcal{L}_{ik}\{F_iF_k\}=\int d1'd2'd2\{W_{ik}(1,2|1',2')F_i(1')F_k(2')-W_{ik}(1',2'|1,2)F_i(1)F_k(2)\}.\qquad(5.8)$$

Let us discuss once more the detailed balance principle for usual particles in the form.

$$\tilde{W}(1',2'|1,2)=\tilde{W}(1,2|1',2').\qquad(5.9)$$

In this form the detailed balance principle describes two-particle collision (a generalization to three-, four-, etc particle collisions is well known). Formula (5.9) means that the ‘‘arriving’’ number in state $(1',2')$ from state (1, 2) is equal to the ‘‘leaving’’ number from state $(1',2')$ to state (1,2). If the total number of arriving particles to some fixed state (1, 2) is equal to the total number of leaving particles from state (1,2) then the new smoothed local balance principle can be formulated as

$$\int W_{ik}(1,2|1',2')d1'd2'=\int W_{ik}(1',2'|1,2)d1'd2',\qquad(5.10)$$

where $i,k=s,b,ph$.

It is not difficult to show that the probabilities of collisions defined in Sec. III satisfy this condition. Thus the probabilities W_{ik} of LE scattering processes in the SG model satisfy the smoothed local balance principle (5.10), while the dissipation parts of probabilities W_{ik}^m satisfy the detailed balance principle (5.9).

The distribution functions *f, B, N* in thermodynamic equilibrium must satisfy the following condition:

$$\mathcal{L}_{ik}\{F_i,F_k\}=0,\qquad(5.11)$$

in accordance with the smoothed local equilibrium principle for each i-k collision integral. It is necessary to emphasize that the smoothed local balance principle (5.10) puts limitations on the probabilities of collisions, on the thermodynamic equilibrium condition (5.11), and on the distribution functions F_i.

Having mentioned the general properties of the collision integrals of LE in the SG equation, we proceed with the examination of each one. From formulas (4.23) and (4.24) it is easy to obtain the following expression for the collision integral for solitons in the low-density case:

$$\mathcal{L}_s\{f_1,F_{2i}\}=-\delta v_s\frac{\partial f_1}{\partial x}+\mathcal{D}_s(v_s)\frac{\partial^2 f_1}{\partial x^2},\qquad(5.12)$$

where renormalization of the soliton velocity δv and the local coefficient of self-diffusion $\mathcal{D}_s(v)$ are given by summing the partial contributions:

$$\delta v_s = \sum_i \delta v_{si}, \quad \mathcal{D}_s(v_s) = \sum_i \mathcal{D}_{si}(v_s). \tag{5.13}$$

Here

$$\delta v_{si} = \int |v_0| \Delta x_{si}(v_{1s}, v_{2i}) F_i d2, \tag{5.14}$$

$$\mathcal{D}_{si} = \frac{1}{2} \int |v_0| [\Delta x_{si}(v_{1s}, v_{2i})]^2 F_i d2. \tag{5.15}$$

The collision integrals for breathers can be analyzed in a similar way, but with a very important difference, connected with the conditions (4.20)–(4.22). The simplest case of a breather ensemble is an ensemble with a distribution function of the form

$$B(x,\varphi,v,\omega_2,t) = B(x,\varphi,v,t)\,\delta(\omega_0 - \omega_2), \tag{5.16}$$

where ω_0 and the density of particles n_i satisfy the condition $\omega_0 m(v) \gg n_i$.

It is possible to consider more general breather distribution functions; e.g.,

$$B = B(x,\varphi,v,t)\, b(\omega),$$

where $b(\omega)$ has a sharp maximum near the point $\omega = \omega_0$. For simplicity only the case (5.16) will be considered. In this case all integrations under ω_2 are trivial.

For breathers the collision integral has the following form:

$$\mathcal{L}_b\{B_1, F_{2i}\} = -\delta v_b \frac{\partial B_1}{\partial x} - \delta\omega_b \frac{\partial B_1}{\partial \varphi} + \mathcal{D}_b \frac{\partial^2 B_1}{\partial x^2} + \mathcal{F}_b \frac{\partial^2 B_1}{\partial \varphi^2} + 2\mathcal{K}_b \frac{\partial^2 B_1}{\partial x \partial \varphi}. \tag{5.17}$$

The first two terms in this formula describe renormalization of the breather velocity δv_b and its internal oscillation frequency $\delta\omega_b$. The last three terms describe self-diffusion in (x,φ) space. As in the soliton case, the quantities δv_b, $\delta\omega_b$, $\mathcal{D}_b$, $\mathcal{F}_b$, and $\mathcal{K}_b$ are sums of the partial contributions

$$\delta v_b = \sum_i (\delta v_b)_i, \quad \delta\omega_b = \sum_i (\omega_b)_i, \quad \mathcal{D}_b = \sum_i (\mathcal{D}_b)_i,$$

$$\mathcal{F}_b = \sum_i (\mathcal{F}_b)_i, \quad \mathcal{K}_b = \sum_i (\mathcal{K}_b)_i, \tag{5.18}$$

where

$$\delta v_{bi} = \int |v_0| \Delta x_{bi}(v_{1b}, v_{2i}) F_{2i} d2, \tag{5.19}$$

$$\delta\omega_{bi} = \int |v_0| \Delta\varphi_{bs}(v_{1b}, v_{2i}) F_{2i} d2, \tag{5.20}$$

$$\mathcal{D}_{bi} = \frac{1}{2} \int |v_0| [\Delta x_{bi}(v_{1b}, v_{2i})]^2 F_{2i} d2, \tag{5.21}$$

$$\mathcal{F}_{bi} = \frac{1}{2} \int |v_0| [\Delta\varphi_{bi}(v_{1b}, v_{2i})]^2 F_{2i} d2, \tag{5.22}$$

$$\mathcal{K}_{bi} = \frac{1}{2} \int |v_0| \Delta x_{bi}(v_{1b}, v_{2i}) \Delta\varphi_{bi}(v_{1b}, v_{2i}) F_{2i} d2. \tag{5.23}$$

VI. KINETIC EQUATIONS AND ENTROPY PRODUCTION

The Boltzmann-type kinetic equations for LE with the collision integrals constructed in the preceding section can be written in the following the standard way as

$$\frac{\partial f}{\partial t} + [v + \delta v_s(v)] \frac{\partial f}{\partial x} = \mathcal{D}_s(v) \frac{\partial^2 f}{\partial x^2}, \tag{6.1}$$

$$\frac{\partial B}{\partial t} + [v + \delta v_b(V)] \frac{\partial B}{\partial x} + [\omega + \delta\omega_b(V)] \frac{\partial B}{\partial \varphi} = \mathcal{D}_b(V) \frac{\partial^2 B}{\partial x^2} + 2\mathcal{K}_b(V) \frac{\partial^2 B}{\partial x \partial \varphi} + \mathcal{F}_b(V) \frac{\partial^2 B}{\partial \varphi^2}, \tag{6.2}$$

$$\frac{\partial N}{\partial t} + [v + \delta v_{ph}(V)] \frac{\partial N}{\partial x} + [\omega + \delta\omega_{ph}(V)] \frac{\partial N}{\partial \varphi} = \mathcal{D}_{ph}(V) \frac{\partial^2 N}{\partial x^2} + 2\mathcal{K}_{ph}(V) \frac{\partial^2 N}{\partial x \partial \varphi} + \mathcal{F}_{ph}(V) \frac{\partial^2 N}{\partial \varphi^2}. \tag{6.3}$$

Here terms from collision integrals describing velocity renormalization have been written on the left-hand sides of Eqs. (6.1)–(6.3). On the right-hand sides of these equations there are only those terms describing dissipative processes leading to homogenization of the distribution functions of LE.

It is necessary to emphasize that the collision integrals in kinetic equations (6.1)–(6.3) are equal to zero in the homogeneous case. Therefore the stationary solution of the kinetic equations have the following form:

$$f = f(v), \quad B = B(v,\omega_2). \tag{6.4}$$

Here $f(v), B(v,\omega_2)$ are arbitrary functions of its arguments.

In other words, the kinetic equations (6.1)–(6.3) describe *homogenization*, i.e., the mixing of the distribution function of LE up to the homogeneous state in real space [for breathers, in (x,φ) space], and demonstrate that chaoticity in momentum space cannot be realized. For chaoticity processes in momentum space it is necessary to exceed the limits of the exactly integrable model, i.e., to take into account the terms destroying the integrability in the Hamiltonian of the system.

Let us now show that homogenization of the distribution function leads to entropy production in the SG localized excitation gas. The entropy of the classical soliton gas and boson gases of breathers and phonons can be defined in the standard way:

$$S=\sum_k S_k, \quad k=s,b,ph, \tag{6.5}$$

$$S_k=-\int F_k \ln(F_k/e)d1. \tag{6.6}$$

The entropy evolution in time is described by the formulas

$$\frac{dS_k}{dt}=-\int \frac{\partial F_k(1)}{\partial t}\ln F_k(1)d1. \tag{6.7}$$

Using kinetic equations (6.1)–(6.3), and the definitions of $\mathcal{D}$, $\mathcal{K}$, and $\mathcal{F}$, one can find that

$$\frac{dS_k}{dt}=\int q_k d1, \tag{6.8}$$

where the source q of the entropy production is

$$q_k=\int \sum_i |v_0(1,2)|_{ki}\frac{F_i}{F_k} \times\left\{[\Delta x(1,2)]_{ki}\frac{\partial F_k(1)}{\partial x}+[\Delta\varphi(1,2)]_{ki}\frac{\partial F_k(1)}{\partial\varphi}\right\}^2 d2. \tag{6.9}$$

It is obvious that the expression is non-negative. This means that Eq. (6.9) proves the Boltzmann entropy production theorem. We would like to emphasize that the entropy production is connected only with inhomogeneity in real space (x,φ). It is easy to see that in the homogeneous case $\partial F/\partial x=\partial F/\partial\varphi=0$, and there no entropy production.

VII. TRANSPORT EQUATIONS

In this section the consequences of kinetic equations (6.1)–(6.3) will be analyzed. Let us emphasize that the homogenization of LE in real space means the homogenization of LE in temperature, concentration, and macroscopic velocity spaces. The local macroscopic temperature $T(x)$ is defined through the local energy $E(x)$ averaged over a distance d_0 around x satisfying the inequality $|\Delta X_i|\ll d_0$. In other words, collisions of LE lead to the creation of diffusion, thermoconductivity, and intrinsic friction processes. It is easy to note that the transport equations will have the form of local conservation laws for each type of LE separately. In the Sine-Gordon system the numbers of solitons, breathers, and phonons are conserved separately. Besides, momentum, energy, and angular velocity $\partial\varphi/\partial t$ (in the breather case) of each LE are conserved in each collision. The transport equation can be written in the following general form:

For solitons:

$$\frac{\partial}{\partial t}n_s\langle a_s\rangle+\frac{\partial}{\partial x}[U_s^r+U_s^m]=0. \tag{7.1}$$

For breathers:

$$\frac{\partial}{\partial t}n_b\langle a_b\rangle+\frac{\partial}{\partial x}[U_b^r+U_b^m]+\frac{\partial}{\partial\varphi}[W_b^r+W_b^m]=0. \tag{7.2}$$

In formulas (7.1) and (7.2) following notations have been used.

For solitons:

$$n_s\langle a_s\rangle=\int a(x,v)f(x,v,t)dv, \tag{7.3}$$

$$U_s^r=\int a(x,v)[v+\delta v]f(x,v,t)dv, \tag{7.4}$$

$$U_s^m=-\frac{\partial}{\partial x}\int a(x,v)\mathcal{D}_s f(x,v,t)dv. \tag{7.5}$$

For breathers:

$$n_b\langle a_b\rangle=\int a(x,v,\varphi,\omega)B(x,v,\varphi,\omega,t)dv\,d\omega, \tag{7.6}$$

$$U_b^r=\int a(x,v,\varphi,\omega)[v+\delta v]B(x,v,\varphi,\omega,t)dv\,d\omega, \tag{7.7}$$

$$U_b^m=-\frac{\partial}{\partial x}\int a(x,v,\varphi,\omega)\mathcal{D}_b B(x,v,\varphi,\omega,t)dv\,d\omega -\frac{\partial}{\partial\varphi}\int a(x,v,\varphi,\omega)\mathcal{K}_b B(x,v,\varphi,\omega,t)dv\,d\omega, \tag{7.8}$$

$$W_b^r=\int a(x,v,\varphi,\omega)[\omega+\delta w]B(x,v,\varphi,\omega,t)dv\,d\omega, \tag{7.9}$$

$$W_b^m=-\frac{\partial}{\partial\varphi}\int a(x,v,\varphi,\omega)\mathcal{F}_b B(x,v,\varphi,\omega,t)dv\,d\omega -\frac{\partial}{\partial x}\int a(x,v,\varphi,\omega)\mathcal{K}_b B(x,v,\varphi,\omega,t)dv\,d\omega. \tag{7.10}$$

Substituting 1, the velocity and energy of LE for the variable a, it is easy to obtain the following transport equations: continuity equations, hydrodynamics equations, and equations for local energy density. Let us write these equations in an explicit form.

Continuity equations $(a=1)$ can be written as

$$\frac{\partial n_s}{\partial t}+\frac{\partial}{\partial x}(j_s^r+j_s^m)=0, \tag{7.11}$$

$$\frac{\partial n_b}{\partial t}+\frac{\partial}{\partial x}(j_b^r+j_b^m)+\frac{\partial}{\partial\varphi}(i_b^r+i_b^m)=0. \tag{7.12}$$

Here the standard notations j_i and i_i have been used for U_i and W_i with $a=1$, respectively.

Hydrodynamics equations can be derived from Eqs. (7.1) and (7.2) with $a=v$ and (v,ω_{2b}) for solitons and breather, respectively, and have the following form:

$$\frac{\partial}{\partial t} n_s u_s + \frac{\partial}{\partial x}(P_s^r + P_s^m) = 0, \tag{7.13}$$

$$\frac{\partial}{\partial t} n_b u_b + \frac{\partial}{\partial x}(P_b^r + P_b^m) + \frac{\partial}{\partial \varphi}(\Pi_b^r + \Pi_b^m) = 0, \tag{7.14}$$

$$\frac{\partial}{\partial t} n_b \omega_b + \frac{\partial}{\partial x}(Q_b^r + Q_b^m) + \frac{\partial}{\partial \varphi}(R_b^r + R_b^m) = 0. \tag{7.15}$$

Here u_s and u_b are the hydrodynamic velocities of solitons and breathers, correspondingly; ω is the hydrodynamic velocity in φ space; P_i^r and P_i^m, $i = s,b$ are the pressures for solitons and breathers; the quantities Π_b^r and Π_m^d are the pressures of breathers due to inhomogeneities in φ space; and the values Q and R are defined by formulas (7.7)–(7.10) with $a = \omega_{2b}$.

The energy transport equations can be derived when $a = E_i$ for solitons and breathers. These equations can be written as

$$\frac{\partial}{\partial t} n_s T_s + \frac{\partial}{\partial x}(U_s^r + U_s^m) = 0, \tag{7.16}$$

$$\frac{\partial}{\partial t} n_b T_b + \frac{\partial}{\partial x}(U_b^r + U_b^m) + \frac{\partial}{\partial \varphi}(W_b^r + W_b^m) = 0. \tag{7.17}$$

The quantities T_i mean the average energies of corresponding LE. When $u_i = \omega_i = 0$, the quantities T_i are the average energies of chaotic motion. U_i and W_i are energy density currents; one can conclude that only U_i^m and W^m are different from 0 when $u_i = \omega_i = 0$.

Let us emphasize the important property of transport equations (7.11)–(7.17). It is easy to see that, for any homogeneous distribution functions,

$$f = f(v,t), \quad B = B(v, \omega_{2b}, t) \tag{7.18}$$

with constant temperatures, hydrodynamic velocities, and chemical potentials μ_i, all dissipative terms in these equations are equal to zero. In other words, there is no energy and momentum exchange between homogeneous gases of solitons and breathers. This special property of Eqs. (7.13)–(7.17) is eliminated by taking into consideration terms in the Hamiltonian that destroy the integrability of the model.

As an example of the explicit calculation of transport coefficients we shall obtain the expression for the self-diffusion coefficients, assuming that the distribution functions of solitons and breathers have following form:

$$f = C_s e^{-m_s v_s^2/2k_B T}, \quad B = C_b e^{-E_b/k_B T} \delta(\omega_2 - \omega_0). \tag{7.19}$$

Let us discuss the concrete expressions for soliton and breather diffusion coefficients due to soliton-soliton, breather-breather, and soliton-breather collisions. For simplicity we will consider the nonrelativistic solitons and breathers only $(v \ll 1)$. This case corresponds to small temperatures $T \ll m$.

The diffusion current of solitons can be presented as

$$j_s^d = \frac{\partial n_s}{\partial x}[n_s D_{ss} + n_b D_{sb}] + \frac{\partial n_b}{\partial x} n_s D_{sb}. \tag{7.20}$$

Using formulas for $(\Delta \varphi)_{ik}$ and $(\Delta x)_{ik}$ from Sec. III, formulas (5.13), (5.18), (7.5), and (7.19) it is possible to calculate both D_{ss} and D_{sb}. We will present here the final results:

$$D_{ss} = (1/4)\,\delta_s^2 (T/\pi M_s)^{1/2} \{[\ln(\gamma M_s/T)]^2 + C\}, \tag{7.21}$$

$$D_{sb} = (1/2)\,\delta_s^2 (2T/\pi \mu_{sb})_{sb}^{1/2} I_{sb}, \tag{7.22}$$

where

$$I_{sb} = \left(\ln \frac{1+\omega_0}{1-\omega_0}\right)^2 \quad \text{if } T/\mu_{sb} \ll 1 - \omega_0^2,$$

$$I_{sb} = [\ln(2\gamma \mu_{sb}/T)]^2 + C \quad \text{if } 1 \gg T/\mu_{sb} \gg 1 - \omega_0^2. \tag{7.23}$$

Here

$$C \approx 1.6, \quad \gamma \approx 1.8.$$

There are two dissipative currents for breathers j_b^d and i_b^d, which can be written as

$$j_b^d = (n_b D_{bb} + n_s D_{bs}) \frac{\partial n_b}{\partial x} + n_b D_{bs} \frac{\partial n_s}{\partial x} + (n_b K_{bb} + n_s K_{bs}) \frac{\partial n_b}{\partial \varphi},$$

$$i_b^d = (n_b F_{bb} + n_s F_{bs}) \frac{\partial n_b}{\partial \varphi} + n_b K_{bs} \frac{\partial n_s}{\partial x} + (n_b K_{bb} + n_s K_{bs}) \frac{\partial n_b}{\partial x}. \tag{7.24}$$

TABLE I. Diffusion coefficients for breather.

ik	$T/\mu_{ik} \ll 1 - \omega_0^2$			$1 \gg T/\mu_{ik} \gg 1 - \omega_0^2$
	J_{ik}^D	J_{ik}^K	J_{ik}^F	J_{ik}^D
bs	$\left[\ln \frac{1+\omega_0}{1-\omega_0}\right]^2$	$\pi \ln \frac{1+\omega_0}{1-\omega_0}$	π^2	$\left[\ln \frac{2\gamma \mu_{bs}}{T}\right]^2 + C$
bb	$\left[\ln \frac{\gamma \omega_0^2 M_b}{(1-\omega_0^2)T}\right]^2 + C$	$\pi \ln \frac{\gamma \omega_0^2 M_b}{(1-\omega_0^2)T}$	π^2	$4\left(\left[\ln \frac{\gamma M_b}{T}\right]^2 + C\right)$

After routine calculations the coefficients D_{ik}, F_{ik}, and K_{ik} can be presented in following general form:

$$D_{bb}=\frac{\delta_b^2}{4}\left(\frac{T}{\pi M_b}\right)^{1/2} J_{bb}^D,$$

$$D_{bs}=\frac{\delta_b\delta_s}{4\sqrt{2\pi}}\left(\frac{T}{\mu_{bs}}\right)^{1/2} J_{bs}^D, \quad D_{ik}=\{D_{ik},F_{ik},K_{ik}\}. \tag{7.25}$$

Here

$$\mu_{ik}=\frac{M_iM_k}{M_i+M_k}, \quad i,k=s,b, \tag{7.26}$$

where M_s and M_b are defined by formulas (2.16).

The expressions for J_{ik}^D are presented in the Table I. It is easy to see that the diffusion of breathers in real space (x space) is much faster than the relaxation on φ.

ACKNOWLEDGMENTS

The authors are grateful to Professor R. N. Gurgi and Professor L. N. Pastur for discussions. Part of this work was supported by INTAS Grant No. 94-3754. I.V.B. and V.G.B. are grateful for the hospitality of the Research Center of Crete/FORTH.

---

[1] E. Fermi, J. Pasta, and S. Ulam, Los Alamos Scientific Report No. LA-1940, 1955 (unpublished).

[2] N. J. Zabusky and M. D. Kruskal, Phys. Rev. Lett. **15**, 240 (1965).

[3] C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. Miura, Phys. Rev. Lett. **19**, 1095 (1967).

[4] V. E. Zakharov, S. V. Manakov, S. P. Novikov, and L. P. Pitaevskii, *Theory of Solitons* (Nauka, Moscow, 1980).

[5] L. D. Faddeev and L. A. Tahtadzhan, *Hamiltonian Methods in the Theory of Solitons* (Nauka, Moscow, 1986).

[6] V. E. Zakharov, Zh. Eksp. Theor. Fiz. **60**, 993 (1971) [Sov. Phys. JETP **33**, 538 (1972).]

[7] T. R. Koehier, A. R. Bishop, J. A. Krumhansl, and J. R. Schrieffer, Solid State Commun. **17**, 1515 (1975).

[8] Y. Wada and J. R. Schrieffer, Phys. Rev. B **18**, 3897 (1978).

[9] N. Theodorakopoulos, Z. Phys. B **33**, 385 (1979).

[10] K. Z. Fesser, Z. Phys. B **39**, 47 (1979).

[11] K. Sasaki and K. Maki, Phys. Rev. B **35**, 257 (1987).

[12] I. V. Baryakhtar, V. G. Baryakhtar, and E. N. Economou, Phys. Lett. A **207**, 67 (1995).

[13] V. G. Baryakhtar, I. V. Baryakhtar, B. A. Ivanov, and A. L. Sukstansky, in *Proceedings of the Second International Symposium on Selected Topics in Statistical Mechanics*, edited by N. N. Bogolubov (JINR, Dubna, 1981), pp. 417–426.

[14] O. M. Braun and Yu. S. Kivshar, Phys. Rep. **306**, 1 (1998).